

THE NON-EUCLIDEAN-EUCLIDEAN ALGORITHM

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Dedicated to the memory of F.W. Gehring

ABSTRACT. In this paper we demonstrate how the geometrically motivated algorithm to determine whether a two generator real Möbius group acting on the Poincaré plane is or is not discrete can be interpreted as a *non-Euclidean Euclidean algorithm*. That is, the algorithm can be viewed as an application of the Euclidean division algorithm to real numbers that represent hyperbolic distances. In the case that the group is discrete and free, the algorithmic procedure also gives a non-Euclidean Euclidean algorithm to find the three shortest curves on the corresponding quotient surface.

1. INTRODUCTION

The problem of determining whether a two generator real Möbius group acting on the Poincaré plane (hyperbolic two-space) is or is not discrete is an old one. If such a group is non-elementary and discrete, it is, of course, a Fuchsian group. There are many approaches to this problem, some of them incomplete. One of the most complete answers is given by the use of an algorithm. The algorithm can be given in a number of different forms [2], including a geometric form and an algebraic form. Revisiting the algorithm has been productive, as the algorithm has been shown to have a number of useful implications, including results about primitives and palindromes in rank two free groups, discreteness criteria for complex Möbius groups acting on hyperbolic three space, and the computational complexity of the discreteness problem [1, 2, 4, 6, 7, 10]. In this paper we revisit the Gilman-Maskit algorithm [8] in the case of a pair of hyperbolic generators A and B with disjoint axes and illustrate that it is a type of *non-Euclidean Euclidean algorithm*. We refer to the Gilman-Maskit algorithm as the GM algorithm.

An algorithm is termed a non-Euclidean Euclidean algorithm if it involves performing Euclidean algorithm type calculations to quantities that are non-Euclidean lengths, in particular to the translation lengths, T_A and T_B , of the isometries when they (or a conjugate pair) act as hyperbolic isometries on the Poincaré plane.

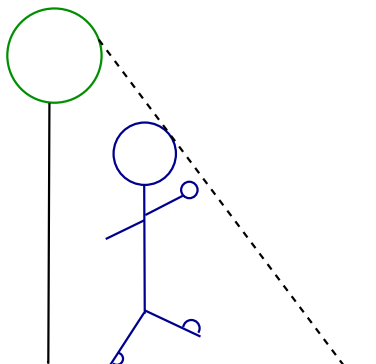


FIGURE 1. In calculus one often asks students to analyze the picture of a person whose shadow is being cast by a lamppost. The analysis can be requested under varying rates (e.g. the rate that the person walks away from the lamppost or the rate that shadow grows) with different quantities shown in the figure known and unknown. Now analyze this situation when the distances are all hyperbolic distances and the rates are also rates of change of the hyperbolic quantities.

The algorithms in [8, 3] can also be viewed as algorithms to find the three shortest geodesic when the group is discrete. The paper [3] addresses the case of intersecting hyperbolic axes. We point out how the concept of the non-Euclidean Euclidean algorithm applies to all types of pairs of isometries acting on hyperbolic two-space as such a shortest length algorithm, even in the case where no algorithm is needed to determine discreteness. This is done in section 6.

A quick heuristic way to understand the term *Non-Euclidean Euclidean algorithm* (an NEE algorithm) is by referring to the description¹ given by Ryan Hoban [9] (see Figure 1).

1.1. The organization of the paper. The main theorems, Theorem 2.3 and its companion Theorem 2.4, are stated in section 2 using a minimal amount of notation and background material. In section 3 we review terminology. This includes subsections on notation, factoring, and the algorithmic paths. It also includes 3.3 where we give a bare-bones description of the algorithm together with figures that illustrate the geometry of some of the cases we need to consider. The proofs of

¹After this interpretation of the algorithm as a NEE algorithm was in place, but before the manuscript was written, the author saw this description in a lecture by Hoban.

the main theorems along with the relevant lemmas and their proofs are in section 4. At the end of this section, section 4, Theorem 2.3 and its companion Theorem 2.4 are combined into a single theorem, Theorem 4.4. In section 5 the analogy is explained and in section 6 elliptic elements are addressed and the result on shortest curves, Corollary 6.1, is stated and proved.

We begin by stating Theorem 2.3, the main result.

2. THE MAIN RESULT

We let G be the group generated by A and B both elements of $PSL(2, \mathbb{R})$ and assume throughout this paper that G is non-elementary. The results of [8, 2, 3, 4] can be summarized as

Theorem 2.1. [8, 1] Geometric Algorithm

Given A and B there is a an integer t and a set of integers $[n_1, \dots, n_t]$ such that replacing the ordered pair (A, B) by the sequence of generators:

$$(A, B) \rightarrow (B^{-1}, A^{-1}B^{n_1}) \rightarrow (B^{-n_1}A, A^{-1}B^{n_1}(A^{-1}B^{n_1})^{n_2}) \rightarrow \dots (C, D)$$

after t steps gives an ordered pair of stopping generators (C, D) and outputs G is

(i) *discrete*

(ii) *not discrete, or*

(iii) *not free.*

We note that in the case that G is not free, the algorithm determines discreteness or non-discreteness and finds stopping generators, but the sequence of integers $[n_1, \dots, n_t]$ must be modified (see section 6). The sequence $[n_1, \dots, n_t]$ was known as the Finonacci/non-Fibonacci sequence of the algorithm [2, 10] and used in the calculation of computational complexity.

Definition 2.2 ([1, 2, 10, 4]). The sequence $[n_1, \dots, n_t]$ is termed the F-sequence or the *Fibonacci sequence* of the algorithm.

We first state the main result in the case that the initial and stopping generators are hyperbolic isometries with disjoint axes as this is the most complicated case of the algorithm and then give extensions allowing parabolics; in section 6 we indicate how this extends to all other type of pairs of initial and stopping generators. We use T_X

and K_X to denote the translation length and multiplier of an element $X \in PSL(2, \mathbb{R})$ or equivalently the translation length and multiplier of a lift of X to $SL(2, \mathbb{R})$ as defined in section 3.1, where $Tr(X)$, the trace of X or a lift is also defined. Note that $K_{X^{-1}} = K_X^{-1}$.

Theorem 2.3. (hyperbolic-hyperbolic initial and stopping generators)
Assume that A and B are a pair of hyperbolics with disjoint axes and the algorithm stops with such a pair.

If one applies the Euclidean algorithm to the non-Euclidean translation lengths of the generators at each step, the output is the F -sequence $[n_1, \dots, n_k]$.

In particular if the multiplier of A is K_A and the multiplier of B is K_B , then

$$n_1 = \left[\frac{(|\log K_A|)/2}{(|\log K_B|)/2} \right]$$

where $[\]$ denotes the greatest integer function and $| \ |$ absolute value, or equivalently if T_X is the translation length of X :

$$n_1 = \left[\frac{T_A/2}{T_B/2} \right].$$

and

$$n_2 = \left[\frac{T_B/2}{T_D/2} \right] \quad \text{where } D = B^{-n_1} A$$

and

$$n_j = \left[\frac{T_{C_j}/2}{T_{D_j}/2} \right]$$

where (C_j, D_j) is the pair of generators at step j , $1 \leq j \leq t$ in Theorem 2.1.

Now we extend this to statements of the theorem if the algorithm encounters parabolics. Note that a parabolic isometry does not have a multiplier. For any X and Y , let $[X, Y]$ be their multiplicative commutator.

Theorem 2.4. (parabolic elements)

Assume that at step $t-1$, D_{t-1} is parabolic with C_{t-1} hyperbolic. Let $A = C_{t-1}$ and $B = D_{t-1}$. Then

$$n_t = \left[\frac{2 \cdot |Tr(A)| - 2}{\sqrt{|Tr([A, B])|}} \right].$$

or equivalently,

$$n_t = \left[\frac{4 \cosh\left(\frac{T_A}{2}\right) - 2}{2 \cosh\left(\frac{T_{[A,B]}}{2}\right)} \right].$$

If C_{t-1} and D_{t-1} are both parabolic, $n_t = 1$.

If one of C_t or D_t is elliptic, the group is either not free or not discrete.

3. PRELIMINARIES

3.1. Notation and Terminology. Let $\tilde{X} \in SL(2, \mathbb{R})/\pm id$ denote any pull-back of X from $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm id$. \tilde{X} acts as a fractional linear transformation on the complex plane and acts as isometries on the upper-half plane model of hyperbolic two-space endowed with the Poincaré metric. Since X and \tilde{X} induce the same isometry, we use them interchangeably. As conjugates of X and \tilde{X} act as isometries on the unit disc model, we move from the upper-half-plane model to the unit disc model according to convenience using X for the element acting in either model.

We remind the reader of the classification of isometries *algebraically* by the absolute value of the trace of the pull back. The matrix \tilde{X} and the isometry induced by X or \tilde{X} is hyperbolic, elliptic or parabolic according to whether $|Tr(\tilde{X})|$, the absolute value of the trace of the matrix, is > 2 , < 2 or $= 2$. If we begin with A and B in $PSL(2, \mathbb{R})$ and choose \tilde{A} and \tilde{B} to have positive trace, then the trace of every element \tilde{X} in the group they generate is determined and we set $Tr(X) = Tr(\tilde{X})$.

If X is hyperbolic, it fixes two points on the boundary of the unit disc and the hyperbolic geodesic connecting the two fixed points known as the axis of X and denoted Ax_X . One of the fixed points is attracting and one is repelling, means that X moves points along the axis of X toward the attracting fixed point and it moves all points on Ax_X a fixed distance in the non-Euclidean metric, its translation length, T_X .

Recall that for X hyperbolic, the transformation \tilde{X} is conjugate to $z \mapsto Kz$ for some real number $K > 0$, known as the *multiplier* and $|Tr(\tilde{X})| = \sqrt{K} + \sqrt{K}^{-1}$. We write K_X for the multiplier of \tilde{X} and note that $T_X = T_{X^{-1}} = |\log K_X|$ via $\cosh \frac{T_X}{2} = \frac{1}{2}Tr(X)$.

3.2. Factorization. Now X can be factored in many ways as the product of any two half-turns. In the hyperbolic case this is as the product of any two half-turns about perpendiculars to Ax_X that intersect Ax_X at a distance of $\frac{T_X}{2}$ apart. If A and B are any two elements of $PSL(2, \mathbb{R})$,

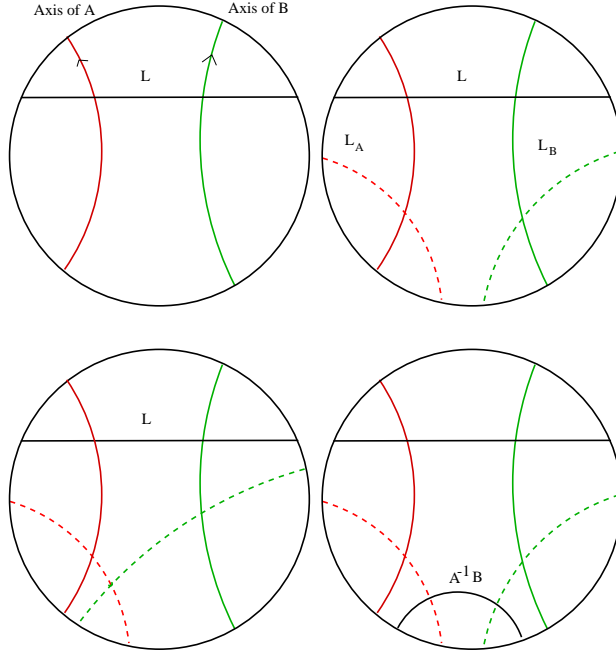


FIGURE 2. Axes of A and B and their common perpendicular L and some possible configurations for the L_A and L_B lines. In the second case the group will be discrete and free. In the third case, $A^{-1}B$ is elliptic and in the last figure the axes of AB^{-1} , the common perpendicular to L_A and L_B , is shown.

their axes have a common perpendicular L and there are lines L_A and L_B such that $A = H_L \circ H_{L_A}$ and $B = H_L \circ H_{L_B}$. A half-turn H_P about a geodesic line P is simply a transformation of order two that fixes the line and interchanges its end points on the boundary of the unit disc (see Figure 2). Elliptics and parabolics still have a so called axis (sometimes a degenerate line) and can again be factored using the common perpendicular.

3.3. The Barebones Algorithm. Assume we begin with A and B hyperbolics with disjoint axes and orient L so that it points from Ax_A to Ax_B when $Tr(A) \geq Tr(B)$ and the attracting fixed points of A and B are both to the left of L .

The algorithm considers the possible configurations for L_A and L_B . For example if these two geodesics intersect then $A^{-1}B$ is either elliptic or parabolic and the algorithm moves up into what is considered to an easier case, where the generating pair is either a hyperbolic-elliptic

pair or a hyperbolic-parabolic pair. If the lines are disjoint and no one separates the other from L , then the three axis L , L_A and L_B bound a region and the algorithm stops indicating that the group is discrete.

In Figure 2 we show some, but not all of the possible configurations for the L_B and L_A lines. We emphasize those that are relevant to our discussion. for example, if $L_A \cap L_B \neq \emptyset$ and lies interior to the unit disc, then $A^{-1}B$ is elliptic so the group even if discrete, is not free. If these lines intersect on the boundary of the disc, then $A^{-1}B$ is parabolic.

If L_B intersects Ax_A and L_A intersects Ax_B , we need to know for how many q does L_{B^q} intersect Ax_A where $B^q = H_L H_{L_{B^q}}$. Analysis of this situation is carried out in Section 4 (see Lemma 4.1). This is termed the *really bad case* and is the focus of most of our attention. Figure 3 (page 13) illustrates the case when $q = 3$ and shows portions of the axes of $B^{-1}A$, $B^{-2}A$, and $B^{-3}A$.

3.4. The algorithmic path. The algorithm considers the different possible types of generating pairs and breaks them down into two cases. One is the case where both generators are hyperbolic and their axes intersect [3]. The other case where the axes of the hyperbolic are disjoint is known as the *intertwining case* [8].

In the intertwining case, the algorithm tests the generators for discreteness and non-discreteness. If neither is found to be true, the algorithm declares the pair indeterminate and produces a *next pair* of generators. The algorithm returns normalized ordered pairs (termed *coherently oriented* pairs) as the next pair so that we always have $Tr(C_j) \geq Tr(D_j)$ at the j -th step. An implementation of the algorithm will begin and end with a pair in any part of the possible paths given below, but the *next pair* will follow the path staying stationary or moving to the right, allowing at most a finite number, n_j , of repetitions of a given pair-type before it moves to the right.

Let H, P and E denote respectively a hyperbolic, parabolic, or elliptic generator. The types of pairs in the intertwining algorithmic path are:

- (1) $H \times H \rightarrow H \times E \rightarrow P \times E \rightarrow E \times E$
- (2) $H \times H \rightarrow H \times P \rightarrow H \times E \rightarrow P \times E \rightarrow E \times E$
- (3) $H \times H \rightarrow H \times P \rightarrow P \times E \rightarrow E \times E$
- (4) $H \times H \rightarrow H \times P \rightarrow P \times P \rightarrow P \times E \rightarrow E \times E$

4. CONFIGURATIONS AND DISTANCES: MAIN LEMMA

We turn our attention to the *really bad case*.

Lemma 4.1. (Main Lemma) *If (A, B) satisfy $\text{Tr}(A) \geq \text{Tr}(B) > 2$, $L_A \cap Ax_B \neq \emptyset$, and $L_B \cap AX_A \neq \emptyset$, then there exists a positive integer n such that*

$$n \frac{(T_B)}{2} \leq \frac{T_A}{2} \leq (n+1) \frac{T_B}{2}.$$

Lemma 4.2. *If (A, B) satisfy $\text{Tr}(A) \geq \text{Tr}(B) > 2$ and $L_A \cap Ax_B = \emptyset$, but $L_B \cap AX_A \neq \emptyset$, then there exists a positive integer n such that*

$$n \frac{(T_B)}{2} \leq \frac{T_A}{2}.$$

Lemma 4.3. *If $\text{Tr}(A) > \text{Tr}(B) = 2$, with $L_B \cap Ax_A \neq \emptyset$, then there is a positive integer q such that $L_{B^q} \cap Ax_A \neq \emptyset$, but $L_{B^{q+1}} \cap L_A = \emptyset$. and $q = \lfloor \frac{2 \cdot |\text{Tr}(A)| - 2}{\sqrt{|\text{Tr}([A, B])|}} \rfloor$. $\text{Tr}(A) = \text{Tr}(B) = 2$, then if $L_A \cap L_B = \emptyset$ the group is discrete and if $L_B \cap L_A \neq \emptyset$, then the group is either not free or not discrete. The F -sequence ends.*

Proof. (Proof of Lemma 4.1.) The distance between L and L_B along the axis of B is $T_B/2$. We consider $L_{B^2}, L_{B^3}, \dots, L_{B^{r-1}}, L_{B^r}$ successive perpendiculars to the axis of B that are each a distance $T_B/2$ from the preceding one along the axis of B and note that $H_{L_{B^r}} \circ H_L = B^r$ and $H_{L_{B^r}} \circ H_{L_{B^s}} = B^{r-s}$. The integer n_j of the F -sequence is the r such that $L_{B^{r-1}}$ separates L and L_A , but L_{B^r} does not.

Further if for a given k L_{B^k} and $L_{B^{k+1}}$ both intersect the axis of A , then the distance between their intersection points along the axis of A is greater than $T_B/2$. Distances along the Axis of B are additive, so that the distance between L and L_{B^k} along the axis of B is $kT_B/2$ and if r is the last integer such that L_{B^r} intersects the axis of A between L and L_A , then $rT_B \leq T_A$.

On the other hand, since L_A intersects Ax_B between L_{B^r} and $L_{B^{r+1}}$, we have $T_A \leq (r+1)T_B$.

Finally, suppose L_{B^r} meets Ax_A at L_A . Then AB^{-r} is elliptic so the group is not discrete and free. If L_A meets Ax_B at L_{B^r} , then again the group has an elliptic element and is not discrete and free. \square

The proof of Lemma 4.2 only requires minor modifications from that of Lemma 4.1 and the integer n_j at step j is thus the $\lfloor \frac{T_A/2}{T_B/2} \rfloor$.

Proof. (Proof of Lemma 4.3.) If the algorithm begins or encounters a parabolic, the final n_t has a different definition. At each algorithmic step (see [8] or [2]) the trace of the hyperbolic in the pair (A, B) where A is hyperbolic and B parabolic is reduced by a fixed amount, which can be computed to be $\sqrt{|\text{Tr}([A, B])|}$. That is, normalize the matrices

by conjugation so that $A(z) = z + \tau$ and $B(z) = \frac{az+b}{bz+a}$ where $a^2 - b^2 = 1$ to see that the amount by which the trace is reduced can be written as $|\tau \cdot b|$. Then n_t is as in the statement of the Lemma. If both A and B are parabolics, then either the product is elliptic and the group is not free or the product is hyperbolic or parabolic and the group is discrete. Thus $n_t = 1$ and can be omitted. \square

We note that if $T_A = T_B$, then after possibly interchanging A and B and replacing A and/or B by its inverse, we may assume that L_B separates L and L_A and we are in the case of one of the first two lemmas.

Proof. (Proof of Theorem 2.3.) Assume that at some point the algorithm returns the pair (A, B) . If L_A and L_B intersect, then the algorithm is stopped and G is either not free and or not discrete. If the three geodesics L , L_A and L_B bound a region, G is discrete. If Jørgensen's inequality is violated, G is not discrete. Otherwise, we are in one of the cases of one of the three lemmas and the result follows. \square

Combining the results we have

Theorem 4.4. *Let $G = \langle A, B \rangle$ and let (C_t, D_t) be the stopping pair for the GM algorithm.*

(1) *Assume that A and B are a pair of hyperbolics with disjoint axes and that the GM discreteness algorithm stops with such a pair. If one applies the Euclidean division algorithm to the non-Euclidean translation lengths of the generators at each step, the output is the F -sequence $[n_1, \dots, n_t]$.*

In particular if the multiplier of A is K_A and the multiplier of B is K_B , then

$$n_1 = \left[\frac{(|\log K_A|)/2}{(|\log K_B|)/2} \right]$$

where $[\]$ denotes the greatest integer function and $| \ |$ absolute value, or equivalently if T_X is the translation length of X :

$$n_1 = \left[\frac{T_A/2}{T_B/2} \right]$$

and

$$n_2 = \left[\frac{T_B/2}{T_D/2} \right] \quad \text{where } D = B^{-n_1} A$$

and

$$n_j = \left[\frac{T_{C_j}/2}{T_{D_j}/2} \right]$$

where (C_j, D_j) is the pair of generators at step j , $1 \leq j \leq t$ in Theorem 2.1.

(2) If at step j (C_j, D_j) is hyperbolic-parabolic pair, then $j = t-1$ and

$$n_t = \left\lceil \frac{2 \cdot |Tr(C_{t-1})| - 2}{\sqrt{|Tr([C_{t-1}, D_{t-1}])|}} \right\rceil$$

or equivalently, setting $A = C_{t-1}$ and $B = D_{t-1}$.

$$n_t = \left\lceil \frac{4 \cosh(\frac{T_A}{2}) - 2}{2 \cosh(\frac{T_{[A,B]}}{2})} \right\rceil.$$

(3) If at step j (C_j, D_j) is a parabolic-parabolic pair, $j = t$ and $n_t = 1$.

For $t \geq 2$, at step $(t-2)$, (C_{t-2}, D_{t-2}) is a hyperbolic-parabolic pair and $n_{t-1} = \left\lceil \frac{2 \cdot |Tr(C_{t-1})| - 2}{\sqrt{|Tr([C_{t-1}, D_{t-1}])|}} \right\rceil$.

(4) If the initial pair is a hyperbolic-parabolic pair, then the F -sequence of length 2 and is $[n_1, n_2]$ where $n_1 = \left\lceil \frac{4 \cosh(\frac{T_A}{2}) - 2}{2 \cosh(\frac{T_{[A,B]}}{2})} \right\rceil$ and $n_2 = 1$. If the initial pair is a parabolic-parabolic pair, the \tilde{F} -sequence is of length 1 with $n_1 = 1$.

5. THE ANALOGY: THE EUCLIDEAN AND THE NON-EUCLIDEAN ALGORITHM

We find the sequence $[n_1, \dots, n_t]$ of the discreteness algorithm by a combination of Euclidean or division algorithm type of computations with hyperbolic lengths and hyperbolic length replacements with the remainder term (in keeping with the trace reducing aspect of the algorithm).

Let $a = (|\log K_A|)/2$ and $b = (|\log K_B|)/2$.

Then these are simultaneously hyperbolic lengths and real numbers. We can do the first step of the Euclidean type algorithm on a and b to obtain (assuming $a > b$):

Step 1. $a = n_1 b + b_1$ where $0 \leq b_1 < b$ and n_1 is a positive integer.

Step 2. At step 2, in a standard Euclidean algorithm, we would normally work with b_1 and b :

BUT we replace b_1 by $T_D/2 = |\log K_D|/2$. (Note: we know from the geometry $T_D/2 \leq b_1$.)

That is, set $D = B^{-n_1} A$ and use $\tilde{b}_1 = T_D/2$.

Subsequent Steps. Next, $b = n_2\tilde{b}_1 + b_2$ where $0 \leq b_2 < b_1$ and n_2 is an integer. Replace b_2 by $\tilde{b}_2 = T_E/2$ where $E = B(A^{-1}B^{n_1})^{n_2}$ and continue. Note that $T_E = T_{E^{-1}}$.

Stop. The geometric proof that the algorithm stops in a finite number of steps (b/c the trace is reduced at each step by at least a minimal amount) says that here after a finite number of steps, we are at a stopping point. This is recognized by a trace that is less than 2 (as the initial traces are taken to be positive). Thus the algorithm stops if either the group contains an elliptic (trace between -2 and 2), the axes bound a region, or Jørgensen's inequality has been violated.

6. INCLUDING ELLIPTICS AND INTERSECTING AXES

In the case that the algorithmic path encounters an elliptic, either the elliptic is of infinite order in which case the group is not discrete or it is of finite order in which case the algorithm proceeds until it reaches a decision about discreteness. The F -sequence needs to be modified (see [11]) because the algorithm uses both hyperbolic distances and angles. Angles are the same for Euclidean and non-Euclidean geometry. The interpretation as a non-Euclidean Euclidean algorithm can be continued, but we do not do so here.

For hyperbolics with intersecting axes, one does not need an algorithm to determine discreteness unless the commutator is elliptic. However, the same rule for replacing generators using the F -sequence will stop and find the shortest generators.

Thus whether we are in the intertwining case or the intersecting axes case, if the group is discrete so that the quotient is a surface, the steps used in the algorithm can be applied and interpreted as an algorithm to find the three shortest geodesics on the quotient (see [4, 5]). This algorithm can be interpreted as a non-Euclidean Euclidean algorithm with the same definitions of the n_i . We have:

Corollary 6.1. (Shortest Curves) (initial generators hyperbolics with intersecting or disjoint axes) *Given $G = \langle A, B \rangle \subset PSL(2, \mathbb{R})$ where A and B are hyperbolic either with disjoint or intersecting axes, there is a non-Euclidean Euclidean Algorithm that finds the generators corresponding to the three shortest curves on the quotient surface when the group is discrete. There is an F -sequence of integers and these integers are calculated by the same formulas as the formulas in Theorem 2.3.*

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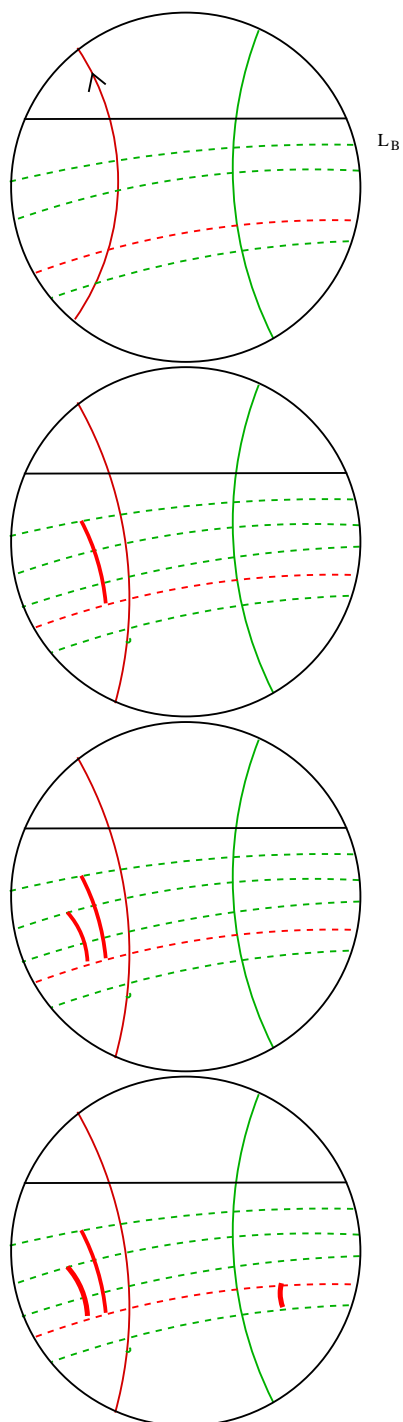


FIGURE 3. A number of L_B lines with L_A intersecting the axis of B . Dotted green lines represent $L_B, L_{B^2}, L_{B^3}, L_{B^4}$ respectively. Axes of $A, B^{-1}A, B^{-2}A$, and $B^{-3}A$ are solid red lines.